$GT_{3\frac{1}{2}}$-spaces, II

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Abstract

In this second part we continue the study of the notion of $GT_{3\frac{1}{2}}$-spaces which we had introduced in part I of this paper. We study here its relation with the L-proximity spaces defined by Katsaras in 1979, the L-uniform spaces defined by Gähler and the first author and others in 1998 and the L-compact spaces defined by Gähler in 1995. The relation between the $GT_{3\frac{1}{2}}$-spaces and the L-topological groups will be studied in a separate paper.

Keywords: L-filters; GT_i-spaces; completely regular spaces; $GT_{3\frac{1}{2}}$-spaces; L-Tychonoff spaces; Initial and final lifts; Initial and final L-topological spaces; L-proximity spaces; L-uniform spaces; L-compact spaces; L-topological groups.

Introduction

In this paper we continue the numbering of sections and begin therefore with Section 6. Throughout this paper we use the same terminology as in part I.

In Section 6 of this paper we shall study the relation between the $GT_{3\frac{1}{2}}$-spaces, which we had introduced and studied in part I of this paper ([7]), and the L-proximity spaces defined by Katsaras in [20]. Using the Urysohn's Lemma, which we had established in part I, and other results which are proved here we show many results joining the completely regular L-topology in our sense and the L-proximity in sense of Katsaras. We show that the L-topology associated with an L-proximity

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is completely regular in our sense. Moreover, we show that every completely regular L-topology is compatible with an L-proximity.

Section 7 is devoted to the study of the relation of the $GT_{3\frac{1}{2}}$-spaces with the L-uniform spaces defined by Gähler and the first author and others in [15]. We had got some results similar to what we had got for L-proximities in Section 6 of this paper. We show that the L-topology associated with an L-uniform structure is completely regular in our sense, and that every completely regular stratified L-topology is compatible with an L-uniform structure, that is, every completely regular stratified L-topology is uniformizable.

The last section is devoted to investigate the relation of the $GT_{3\frac{1}{2}}$-spaces with the L-compact spaces defined by Gähler in [13], which is called G-compact spaces. We show also here some results joining the $GT_{3\frac{1}{2}}$-spaces and the G-compact spaces. We show that the L-unit interval $(I_L, \Im)$ and that the L-cube, defined as a product of L-unit intervals are G-compact GT_2-spaces and consequently GT_4-spaces and hence they are $GT_{3\frac{1}{2}}$-spaces. We show also that a G-compact space is a GT_2-space if and only if it is a $GT_{3\frac{1}{2}}$-space. If τ_1 and τ_2 are L-topologies on a set X with τ_1 is finer than τ_2, and (X, τ_1) is a G-compact space and (X, τ_2) is a $GT_{3\frac{1}{2}}$-space, then we prove that τ_1 is equivalent to τ_2.

Moreover, we show that an L-topological space (X, τ) is a $GT_{3\frac{1}{2}}$-space if and only if it is homeomorphic to a subspace of an L-cube if and only if it is homeomorphic to a subspace of a G-compact GT_2-space if and only if it is homeomorphic to a subspace of a GT_4-space.

6. The relation between the $GT_{3\frac{1}{2}}$-spaces and the L- proximity spaces

In this section we are going to study and prove some results joining the L-proximity spaces defined by Katsaras in [20] and the $GT_{3\frac{1}{2}}$-spaces which we had introduced in [7].

We recall and use here all notations and definitions given in [7].

Now, we shall prove that the L-topology τ_δ associated with an L-proximity δ is completely regular.

Proposition 6.1 *If δ is an L-proximity on X, then τ_δ is completely regular.*

Proof. Let $x \in X$ and F be a closed subset of X with $x \notin F$. Since $\chi_{F'}$ is a τ_δ-neighborhood of x, then $x_1 \,\overline{\delta}\, \chi_F$. On account of Proposition 2.5, we get that x_1 and χ_F are separated by a proximally continuous function $f : (X, \delta) \to (I_L, \delta^*)$ which is also, by means of Proposition 2.3, $(\tau_\delta, \Im_{\delta^*})$-continuous. Hence, τ_δ is completely regular. $\Box$

To examine for a given L-topology τ on a set X, when an L-proximity on X compatible with τ exists, we need the following proposition. It will be shown that this happens if and only if τ is completely regular.

Proposition 6.2 *Let (X, τ) be an L-topological space and let Φ be an L-function family of all $(\tau_k, \Im)$-continuous functions, $f_k : (X, \tau_k) \to (I_L, \Im)$, $k \in K$ and K any class. Then (X, τ) is a completely regular space* if and only if *τ coincides with the coarsest L-topology on X for which each member of Φ is $(\tau_k, \Im)$-continuous.*

Proof. Let (X, τ) be a completely regular space. Then there is a $(\tau, \Im)$-continuous mapping. If σ is the coarsest L-topology on X with respect to which each member of Φ is $(\tau_k, \Im)$-continuous, then τ is one of these τ_k and hence we have that $\sigma \subseteq \tau$ holds.

Let $x \in X$, $\mu \in \tau$ and $x_1 \leq \mu$. Then there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in s_0\mu'$. From the hypothesis that σ is the coarsest L-topology on X for which each member of Φ is L-continuous, we get that f is $(\sigma, \Im)$-continuous, and it follows that $\lambda = f^{-1}(R_{\frac{1}{2}}) \in \sigma$ such that $\lambda(x) = R_{\frac{1}{2}}(f(x)) = R_{\frac{1}{2}}(\overline{1}) = 1$ and $\lambda(y) = R_{\frac{1}{2}}(f(y)) = R_{\frac{1}{2}}(\overline{0}) = 0$ for all

$y_1 \leq \mu'$. This means that $x_1 \leq \lambda$ and $\mu' \leq \lambda'$, that is, $x_1 \leq \lambda$ and $\lambda \in \sigma$ with $x_1 \leq \lambda \leq \mu$. Hence, $\mu \in \sigma$ and then $\tau \subseteq \sigma$. Thus τ coincides with σ.

Conversely; assume that τ coincides with the coarsest L-topology on X for which each member of Φ is $(\tau_k, \Im)$-continuous, then each member of Φ is $(\tau, \Im)$-continuous. Since

$$B \;=\; \{R_\eta \circ f \;\mid\; f \in \Phi, \eta \in I\} \;\cup\; \{R^\eta \circ f \;\mid\; f \in \Phi, \eta \in I\} \cup \{\,\overline{0}, \overline{1}\,\}$$

is a base for τ, then defining $g : X \to I_L$ by $g(y)(s) = 1 - f(y)(1-s)$ for all $f \in \Phi$, $s \in I_{01}$, $y \in X$, we get that $g^{-1}(R_\eta) = f^{-1}(R^{1-\eta})$ and $g^{-1}(R^\eta) = f^{-1}(R_{1-\eta})$ and hence, g is $(\tau, \Im)$-continuous and moreover

$$B = \{f^{-1}(R_\eta) \;\mid\; f \in \Phi, \eta \in I_{01}\} \cup \{\,\overline{0}, \overline{1}\,\}$$

.

Now, let $H \in B$, $x \in X$ with $x \in H$. Then, there exists $f \in \Phi$ and $t_0 \in I_{01}$, such that $\chi_H = f^{-1}(R_{t_0})$. For each $y \in X$, define $g(y) : I \to L$ by $g(y)(t) = f(y)(t_0 + t(1-t_0))$, then $g^{-1}(R_t) = f^{-1}(R_{t_0+t(1-t_0)})$ and $g^{-1}(R^t) = f^{-1}(R^{t_0+t(1-t_0)})$ and thus $g : X \to I_L$ is $(\tau, \Im)$-continuous. Also, $R_0(g(y)) = R_{t_0}(f(y)) = f^{-1}(R_{t_0})(y) = \chi_H(y)$ for all $y \in X$, $t_0 \in I_{01}$, $f \in \Phi$, that is, $R_0(g(x)) = 1$ and $R_0(g(y)) = 0$ for all y with $y \in H'$, which means that $g(y) = \overline{0}$ for all $y \in H'$ and $g(x)(t) = 1$ for some $t \in I_{01}$, and thus there exists $r \in I_{01}$ such that $R^r(g(x)) = \bigwedge_{k \geq r} (g(x)(k))' = 1$. Defining $h : X \to I_L$ by $h(z)(s) = g(z)(rs)$ for all $z \in X$, $s \in I_{01}$, we get h is $(\tau, \Im)$-continuous and $R_0(h(y)) = R_0(g(y)) = 0$ for all $y \in H'$ and $R_0(h(x)) = R_0(g(x)) = 1$, and $R^1(h(x)) = R^r(g(x)) = 1$, that is, $h(x) = \overline{1}$ and $h(y) = \overline{0}$ for all $y \in H'$. Hence, from Theorem 2.1, the space (X, τ) is completely regular. $\square$

From that every $GT_{3\frac{1}{2}}$-space is a GT_1-space and from Proposition 2.6, we can deduce the following result.

Corollary 6.1 If (X, τ) is a $GT_{3\frac{1}{2}}$-space and Φ is an L-function family of all L-continuous functions $f : (X, \tau) \to (I_L, \Im)$, then any two distinct points in X are Φ-separated.

We should notice that Proposition 2.4 gives us another L-proximity δ which is compatible with these GT_4-topologies τ from Proposition 2.7. Now, we have the following important result which shows that there is an L-proximity compatible with the completely regular L-topologies.

Proposition 6.3 *Let (X, τ) be a completely regular space and Φ an L-function family of all $(\tau, \Im)$-continuous functions. Then the binary relation δ on L^X, defined by*

$$f \,\overline{\delta}\, g \Longleftrightarrow f,\ g \text{ are } \Phi\text{-separated},$$

for all $f, g \in L^X$, is an L-proximity on X compatible with τ, that is, $\tau_\delta = \tau$.

Proof. Let $f \,\overline{\delta}\, g$. Then there exists a function $\lambda \in \Phi$ such that $\lambda(x) = \overline{1}$ for all $x_1 \leq f$ and $\lambda(y) = \overline{0}$ for all $y_1 \leq g$. Defining $\mu : X \to I_L$ by $\mu(x)(s) = 1 - \lambda(x)(1-s)$ for all $x \in X$ and all $s \in I$, then μ is $(\tau, \Im)$-continuous and $\mu(x)(s) = 0$ for all $x_1 \leq f$ and $\mu(y)(s) = 1$ for all $y_1 \leq g$, that is, $g \,\overline{\delta}\, f$ and hence the condition (P1) of an L-proximity is fulfilled.

Obviously, $(f \vee g) \,\overline{\delta}\, h$ implies $f \,\overline{\delta}\, h$ and $g \,\overline{\delta}\, h$. On the other hand $f \,\overline{\delta}\, h$ and $g \,\overline{\delta}\, h$ means that there are $\lambda, \mu \in \Phi$ such that $\lambda(x) = \overline{1}$ for all $x_1 \leq f$ and $\lambda(y) = \overline{0}$ for all $y_1 \leq h$, and $\mu(x) = \overline{1}$ for all $x_1 \leq g$ and $\mu(y) = \overline{0}$ for all $y_1 \leq h$. If we take $\nu : X \to I_L$ with $\nu(x)(s) = \max\{\lambda(x)(s), \mu(x)(s)\}$ for all $x \in X$ and all $s \in I$, then ν is $(\tau, \Im)$-continuous and $\nu(x) = \overline{1}$ for all $x_1 \leq f$ or $x_1 \leq g$ and $\nu(y) = \overline{0}$ for all $y_1 \leq h$. Hence $(f \vee g) \,\overline{\delta}\, h$ and thus (P2) is fulfilled.

Defining $h : X \to I_L$ by $h(x)(s) = 0$ for all $x \in X$ and all $s \in I$, we get $h(x) = \overline{0}$ for all $x \in X$ and h is $(\tau, \Im)$-continuous. So, we can say that $h(x) = \overline{1}$ for any $x_1 \leq \overline{0}$ and $h(y) = \overline{0}$ for any $g \in L^X$. That is, $\overline{0}$ and g are Φ-separated for all $g \in L^X$ and hence $f = \overline{0}$ or $g = \overline{0}$ implies $f \,\overline{\delta}\, g$. Thus (P3) is fulfilled.

It is clear from the definition of δ that $f \,\overline{\delta}\, g$ implies $f \leq g'$ and hence (P4) holds.

Let $f \,\overline{\delta}\, g$ and let $\mu \in \Phi$ such that $\mu(x) = \overline{1}$ for all $x_1 \leq f$ and $\mu(y) = \overline{0}$ for all

$y_1 \leq g$. The functions $\lambda_1, \lambda_2 : X \to I_L$ defined by

$$\lambda_1(x)(s) = \mu(x)(\frac{1+s}{2})$$

and

$$\lambda_2(x)(s) = \mu(x)(\frac{s}{2})$$

for all $x \in X$ and all $s \in I_{01}$, are $(\tau, \Im)$-continuous and also $\lambda_1(x)(s) = 1$ for all $x_1 \leq f$ and $\lambda_2(x)(s) = 0$ for all $x_1 \leq g$, but

$$R^1(\lambda_1(x)) = R^1(\mu(x)) \geq R^{\frac{1}{2}}(\mu(x)) \geq R_{\frac{1}{2}}(\mu(x)) = R_0(\lambda_1(x))$$

and

$$R^1(\lambda_2(x)) = R^{\frac{1}{2}}(\mu(x)) \leq R_0(\mu(x)) = R_0(\lambda_2(x)).$$

So, if we put $h = (\mu^{-1}(R^{\frac{1}{2}}))'$ where $\mu^{-1}(R^{\frac{1}{2}}) \in L^X$, we get

$$R_0(\lambda_1(x)) \leq h'(x) \leq R_0(\lambda_2(x)),$$

and then $\lambda_1(x)(s) = 0$ for all $x_1 \leq h$ and $s \in I_{01}$ and $\lambda_2(x)(s) = 1$ for all $x_1 \leq h'$ and $s \in I_{01}$. That is, $\lambda_1(x) = \overline{1}$ for all $x_1 \leq f$ and $\lambda_1(y) = \overline{0}$ for all $y_1 \leq h$, and moreover $\lambda_2(x) = \overline{1}$ for all $x_1 \leq h'$ and $\lambda_2(y) = \overline{0}$ for all $y_1 \leq g$. Hence, $f \,\overline{\delta}\, h$ and $h' \,\overline{\delta}\, g$ and (P5) is fulfilled. Thus δ is an L-proximity on X.

Now, let $g \in \tau'_\delta$ and $x \in X$ with $g'(x) = 1$. Since $g(y) = \mathrm{cl}_\delta g(y) = \bigwedge_{g \overline{\delta} h'} h(y)$, then there exists $h \in L^X$ with $g \,\overline{\delta}\, h'$ such that $h(x) = 0$, and $g \,\overline{\delta}\, h'$ implies there exists $f \in \Phi$ such that $f(x) = \overline{1}$ for all $x_1 \leq g$ and $f(y) = \overline{0}$ for all $y_1 \leq h'$. Taking $\mu = f^{-1}(R^{\frac{1}{2}})$, we get $\mu(y) = R^{\frac{1}{2}}(f(y)) \geq (R_0(f(y)))' = 1$ for all $y_1 \leq g'$ where $g \,\overline{\delta}\, h'$ implies $h' \leq g'$, and moreover $\mu(y) = R^{\frac{1}{2}}(f(y)) \leq R^1(f(y)) \leq g'(y)$ for all $y \in X$. That is, $\mu \in \tau$ with $x_1 \leq \mu$ and $\mu \leq g'$ which means $g' \in \tau$ and then $g \in \tau'$ and $\tau_\delta \subseteq \tau$.

Conversely; let $g \in \tau'$ and $g \neq \mathrm{cl}_\delta g$, that is, there is $x \in X$ with $\mathrm{cl}_\delta g(x) > 0$ and $g(x) = 0$. Since $x \in s_0 g' \in \tau$ and (X, τ) is a completely regular space, then there exists $f \in \Phi$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in s_0 g$. Let $\mu \in L^X$ be defined by $\mu(y) = (R^1(f(y)))' = \bigvee_{t\geq 1} f(y)(t)$ for all $y \in X$, then $\mu(y) \leq R_0(f(y)) \leq g'(y)$ for all $y \in X$, which means that $\bigvee_{t\geq 1} f(x)(t) = 1$ for all $x_1 \leq \mu$ and $\bigvee_{s>0} f(y)(s) = 0$ for all $y_1 \leq g$, that is, $f(x) = \overline{1}$ for all $x_1 \leq \mu$ and $f(y) = \overline{0}$ for all $y_1 \leq g$, and this means that μ, g are Φ-separated, and so $\mu \,\overline{\delta}\, g$. Now, $\mu \,\overline{\delta}\, g$ implies

$$\mathrm{cl}_\delta g(x) = \bigwedge_{g \,\overline{\delta}\, \lambda'} \lambda(x) \leq \mu'(x) = R^1(f(x)) = 0,$$

that is, $\mathrm{cl}_\delta g(x) = 0$ which is a contradiction and then $g \in \tau'_\delta$. Therefore, $\tau = \tau_\delta$, that is, δ is compatible with τ. $\square$

Example of an L-proximity. Now, we introduce an example of an L-proximity and we show that it induces a completely regular L-topology.

Example 6.1 Let $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then, by means of Example 2.1, (X, τ) is a $GT_{3\frac{1}{2}}$-space. Let δ be the binary relation on L^X defined by

$$f \,\overline{\delta}\, g \iff \begin{array}{l} \text{there is a } (\tau, \Im)\text{-continuous function } h : (X, \tau) \to (I_L, \Im) \text{ such that} \\ h(x) = \overline{1} \text{ for all } x \in X \text{ with } x_1 \leq f \text{ and } h(y) = \overline{0} \text{ for all } y_1 \leq g \end{array}$$

for all $f, g \in L^X$. δ is, by means of Proposition 6.3, an L-proximity on X compatible with τ, that is, the L-topology τ_δ associated with δ is completely regular.

The following result also goes well.

Proposition 6.4 *Let (X, δ) be an L-proximity space and $f, g \in L^X$ with $f \,\overline{\delta}\, g$, and let Φ be the family of those L-proximally continuous functions of (X, δ) into the L-proximity space (I_L, ρ). Then f and g are separated by a $(\tau_\delta, \Im_\rho)$-continuous function from X into I_L.*

Proof. From (2.1), Lemma 2.1 and Remark 2.1 we can deduce that f and g are Φ-separated, and therefore, by means of Proposition 2.3, they are separated by a $(\tau_\delta, \Im_\rho)$-continuous function. $\square$

If δ_1 and δ_2 are two L-proximities on a set X, then δ_1 is *finer than* δ_2 or δ_2 is *coarser than* δ_1, provided

$$f \,\overline{\delta_2}\, g \text{ implies } f \,\overline{\delta_1}\, g$$

for all $f, g \in L^X$ ([14, 20]).

From the last proposition we prove this result.

Proposition 6.5 *Let τ_1, τ_2 be two completely regular L-topologies on a set X. Let δ_1 be an L-proximity compatible with τ_1 and δ_2 the L-proximity defined by*

$$f \,\overline{\delta_2}\, g \Longleftrightarrow f,\ g \text{ are } \Phi\text{-separated in } (X, \tau_2).$$

Then, τ_2 is finer than τ_1 implies δ_2 is finer than δ_1.

Proof. Suppose that $f \,\overline{\delta_1}\, g$. By Proposition 6.3 there exists a $(\tau_1, \Im)$-continuous function $h : X \to I_L$ such that $h(x) = \overline{1}$ for all $x_1 \leq f$ and $h(y) = \overline{0}$ for all $y_1 \leq g$. Since $\tau_1 \subseteq \tau_2$, then h is $(\tau_2, \Im)$-continuous and from the definition of δ_2 we get that $f \,\overline{\delta_2}\, g$. Hence, δ_2 is finer than δ_1. $\square$

7. The relation between the $GT_{3\frac{1}{2}}$-spaces and the L-uniform spaces

This section is devoted to study the relation of the L-uniform spaces defined in [15] with the $GT_{3\frac{1}{2}}$-spaces.

L-uniform structures. Let $\mathcal{U}$ be an L-filter on $X \times X$. The *inverse* $\mathcal{U}^{-1}$ of $\mathcal{U}$ is an L-filter on $X \times X$ defined by $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ for all $u \in L^{X \times X}$, where u^{-1} is the inverse of u defined by: $u^{-1}(x, y) = u(y, x)$ for all $x, y \in X$ ([15]).

For each pair (x, y) of elements x, y of X, the mapping $(x, y)^{\bullet} : L^{X \times X} \to L$ defined by $(x, y)^{\bullet}(u) = u(x, y)$ for all $u \in L^{X \times X}$ is a homogeneous L-filter on $X \times X$.

Let $\mathcal{U}$ and $\mathcal{V}$ be L-filters on $X \times X$ such that $(x,y)^{\bullet} \leq \mathcal{U}$ and $(y,z)^{\bullet} \leq \mathcal{V}$ hold for some $x, y, z \in X$. Then the *composition* $\mathcal{V} \circ \mathcal{U}$ of $\mathcal{U}$ and $\mathcal{V}$ is the L-filter on $X \times X$ defined by

$$(\mathcal{V} \circ \mathcal{U})(w) \ = \bigvee_{v \circ u \leq w} (\mathcal{U}(u) \wedge \mathcal{V}(v))$$

for all $w \in L^{X \times X}$, where $u, v, v \circ u \in L^{X \times X}$ and $(v \circ u)(x,y) = \bigvee_{z \in X} (\, u(x,z) \wedge v(z,y))$ for all $x, y \in X$ ([15]).

By an *L-uniform structure* $\mathcal{U}$ on a set X ([15]) we mean an L-filter on $X \times X$ such that:

(U1) $(x,x)^{\bullet} \leq \mathcal{U}$ for all $x \in X$.

(U2) $\mathcal{U} = \mathcal{U}^{-1}$.

(U3) $\mathcal{U} \circ \mathcal{U} \leq \mathcal{U}$.

An L-uniform structure $\mathcal{U}$ on X is called a *homogeneous L-uniform structure* if it is a homogeneous L-filter on $X \times X$. A set X equipped with an L-uniform structure (homogeneous L-uniform structure) $\mathcal{U}$ is called an *L-uniform space* (*homogeneous L-uniform space*).

If $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are L-uniform spaces, then the mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is said to be *L-uniformly continuous* provided

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \ \leq \ \mathcal{V}$$

holds.

Let $\mathcal{U}$ be an L-filter on $X \times X$ such that $(x,x)^{\bullet} \leq \mathcal{U}$ holds for all $x \in X$, and let $\mathcal{M}$ be an L-filter on X. Then the mapping $\mathcal{U}[\mathcal{M}] : L^X \to L$, defined by

$$\mathcal{U}[\mathcal{M}](f) \ = \bigvee_{u[g] \leq f} (\mathcal{U}(u) \wedge \mathcal{M}(g))$$

for all $f \in L^X$, is an L-filter on X, called the image of $\mathcal{M}$ with respect to $\mathcal{U}$ ([15]), where $u \in L^{X \times X}$ and $g, u[g] \in L^X$ such that:

$$u[g](x) = \bigvee_{y \in X} (g(y) \wedge u(y,x)).$$

To each L-uniform structure $\mathcal{U}$ on X is associated a stratified L-topology $\tau_{\mathcal{U}}$. The related interior operator $\mathrm{int}_{\mathcal{U}}$ is given by ([15]):

$$(\mathrm{int}_{\mathcal{U}} f)(x) \;=\; \mathcal{U}[\dot{x}](f) \tag{7.1}$$

for all $x \in X, f \in L^X$. An L-set f is called a *$\tau_{\mathcal{U}}$-neighborhood* of $x \in X$ provided $\mathcal{U}[\dot{x}] \leq \dot{f}$.

Proposition 7.1 [15] *Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be an L-uniformly continuous mapping between L-uniform spaces. Then the mapping $f : (X,\tau_{\mathcal{U}}) \to (Y,\tau_{\mathcal{V}})$ between the associated L-topological spaces is $(\tau_{\mathcal{U}},\tau_{\mathcal{V}})$-continuous.*

From (1.2) and (7.1) we have the following

$$\mathcal{U}[\dot{x}] \;=\; \mathcal{N}(x) \;\text{ and }\; \mathcal{U}[\dot{f}] \;=\; \mathcal{N}(f) \tag{7.2}$$

for all $x \in X,\ f \in L^X$, where $\mathcal{N}(x)$ and $\mathcal{N}(f)$ are the L-neighborhood filters of the space $(X,\tau_{\mathcal{U}})$ at x and f, respectively.

An L-proximity δ on a set X is called *stratified* if $\overline{\alpha}\,\overline{\delta}\,\overline{\alpha}'$ for all $\alpha \in L$ ([14, 20]).

We have the following result.

Proposition 7.2 [6] *Let (X,τ) be an L-topological space. Then the binary relation δ on L^X which is defined by*

$$g\,\overline{\delta}\,f \quad \text{if and only if } \mathcal{N}(g) \leq \dot{f}',$$

for all $f,g \in L^X$, is an L-proximity on X, where $\leq$ is the finer relation between L-filters and $\mathcal{N}(g)$ is the L-neighborhood filter of (X,τ) at g.

From (7.2) and Proposition 7.2 we shall deduce the following important result.

Proposition 7.3 *For an L-uniform structure $\mathcal{U}$ on X, we get that the binary relation $\delta_{\mathcal{U}}$ on L^X defined by*

$$f\,\overline{\delta_{\mathcal{U}}}\,g \Leftrightarrow \mathcal{U}[\dot{f}] \leq \dot{g}', \tag{7.3}$$

for all $f, g \in L^X$, is a stratified L-proximity on X, and moreover both of the L-uniform structure $\mathcal{U}$ and the induced stratified L-proximity $\delta_{\mathcal{U}}$ are associated with the same stratified L-topology $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$.

Proof. From (7.2) and Proposition 7.2, we get that $\delta_{\mathcal{U}}$, defined by (7.3), is an L-proximity on X. Since $\overline{\alpha} \in \tau_{\mathcal{U}}$ for all $\alpha \in L$, then $\mathcal{U}[\dot{\overline{\alpha}}] \leq \dot{\overline{\alpha}}$ for all $\alpha \in L$, and thus $\overline{\alpha}\, \overline{\delta_{\mathcal{U}}}\, \overline{\alpha}'$ for all $\alpha \in L$. That is, $\delta_{\mathcal{U}}$ is a stratified L-proximity on X. From (7.3) we get that $x_1\, \overline{\delta_{\mathcal{U}}}\, f' \Longleftrightarrow \mathcal{U}[\dot{x}] \leq \dot{f}$, that is, f is a $\tau_{\delta_{\mathcal{U}}}$-neighborhood of x if and only if it is a $\tau_{\mathcal{U}}$-neighborhood of x. Hence both of $\mathcal{U}$ and $\delta_{\mathcal{U}}$ are associated with the same stratified L-topology $\tau_{\mathcal{U}} = \tau_{\delta_{\mathcal{U}}}$. $\square$

We shall use the following result.

Proposition 7.4 [15] *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be two L-uniform spaces. Then $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is L-uniformly continuous* if and only if *$f : (X, \delta_{\mathcal{U}}) \to (Y, \delta_{\mathcal{V}})$ between the associated stratified L-proximity spaces is $(\delta_{\mathcal{U}}, \delta_{\mathcal{V}})$-continuous.*

From Proposition 2.5 and from Propositions 7.3 and 7.4, we can deduce the following.

Proposition 7.5 *Let $F, G \in P(X)$ with $\mathcal{U}[\dot{F}] = \mathcal{U}[\dot{\chi_F}] \leq \dot{\chi_{G'}} = \dot{G}'$ in the L-uniform space $(X,\mathcal{U})$ and let Φ be the family of those L-uniformly continuous functions $h : (X,\mathcal{U}) \to (I_L,\mathcal{U}^*)$ for which $x \in X$ implies $\overline{0} \leq h(x) \leq \overline{1}$. Then χ_F and χ_G are Φ-separable.*

Proof. From Proposition 7.3, we have $\chi_F\, \overline{\delta_{\mathcal{U}}}\, \chi_G$, and from Proposition 2.5, we get that χ_F, χ_G are separated by an L-proximally continuous mapping $f : (X, \delta_{\mathcal{U}}) \to (I_L, \delta_{\mathcal{U}^*})$. Proposition 7.4 implies that $f : (X,\mathcal{U}) \to (I_L,\mathcal{U}^*)$ is then L-uniformly continuous. $\square$

Now, we shall prove that the stratified L-topology associated with an L-uniform structure is completely regular.

Proposition 7.6 *If $\mathcal{U}$ is an L-uniform structure on X and $\tau_{\mathcal{U}}$ the L-topology associated with $\mathcal{U}$, then $(X, \tau_{\mathcal{U}})$ is a completely regular space.*

Proof. Let $x \in X$ and $F \in \tau'_{\mathcal{U}}$ with $x \notin F$. Since $\chi_{F'}$ is a $\tau_{\mathcal{U}}$-neighborhood of x, that is, $\mathcal{U}[\dot{x}] = \mathcal{N}(x) \leq \dot{F'}$. On account of Proposition 7.5, we get that x_1 and χ_F are separated by an L-uniformly continuous function $f : (X,\mathcal{U}) \to (I_L,\mathcal{U}^*)$ which is also, by means of Proposition 7.1, $(\tau_{\mathcal{U}}, \Im_{\mathcal{U}^*})$-continuous. That is, $(X, \tau_{\mathcal{U}})$ is a completely regular space. □

Example of an L-uniform structure. In the following we give an example of an L-uniform structure and we show that it induces a completely regular L-topology.

Example 7.1 The L-metrics in sense of S. Gähler and W. Gähler ([11]) canonically generate homogeneous L-uniform structures as follows: For each L-metric ϱ on a set X, the mapping $\mathcal{U}_\varrho : L^{X\times X} \to L$, defined by

$$\mathcal{U}_\varrho(u) \;=\; \bigvee_{\varepsilon_{\alpha,\,\delta}\circ\varrho\leq u,\,0<\delta} \alpha$$

for all $u \in L^{X\times X}$, is a homogeneous L-uniform structure on X and moreover $\tau_{\mathcal{U}_\varrho} = \tau_\varrho$ (cf. [15]). From Propositions 2.8 and 2.9 we get that τ_ϱ and hence $\tau_{\mathcal{U}_\varrho}$ is a completely regular stratified topology.

The L-uniform structures can be characterized by means of families of prefilters on $X \times X$ as follows.

Proposition 7.7 [15] *There is a one - to - one correspondence between the L-uniform structures $\mathcal{U}$ on X and the families $(\mathcal{U}_\alpha)_{\alpha\in L_0}$ of prefilters on $X \times X$ which fulfill the following conditions:*

(u1) $0 < \beta \leq \alpha$ *implies* $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$.

(u2) *For each* $\alpha \in L_0$ *with* $\bigvee_{0<\beta<\alpha} \beta = \alpha$ *we have* $\mathcal{U}_\alpha = \bigcap_{0<\beta<\alpha} \mathcal{U}_\beta$.

(u3) *For all* $\alpha \in L_0$, $u \in \mathcal{U}_\alpha$ *and* $x \in X$ *we have* $\alpha \leq u(x, x)$.

(u4) $u \in \mathcal{U}_\alpha$ *implies* $u^{-1} \in \mathcal{U}_\alpha$ *for all* $\alpha \in L_0$.

(u5) *For each* $\alpha \in L_0$ *and each* $u \in \mathcal{U}_\alpha$, *we have* $\alpha \leq \bigvee_{v \in \mathcal{U}_\beta,\, v \circ v \leq u} \beta$.

This correspondence is given by

$$\mathcal{U}_\alpha = \alpha\text{-pr}\,\mathcal{U} \text{ for all } \alpha \in L_0 \text{ and } \mathcal{U}(u) = \bigvee_{v \in \mathcal{U}_\alpha,\, v \leq u} \alpha$$

for all $u \in L^{X \times X}$, *where* $\alpha\text{-pr}\,\mathcal{U} = \{u \in L^{X \times X} \mid \mathcal{U}(u) \geq \alpha\}$.

An L-topogenous order (structure) $\ll$ is called *perfect* ([22]) if for each family $(f_i)_{i \in I}$ of L-subsets of X with $f_i \ll g$ for all $i \in I$ it follows $\bigvee_{i \in I} f_i \ll g$.

Proposition 7.8 [22] *There is a one - to - one correspondence between the perfect L-topogenous structures $\ll$ on a set X and the L-topologies τ on X. This correspondence is given by*

$$f \ll g \;\Leftrightarrow\; f \leq k \leq g \text{ for some } k \in \tau$$

for all $f, g \in L^X$ *and*

$$\tau \;=\; \{\, f \in L^X \mid f \ll f \,\}.$$

Let (X, τ) be a stratified L-topological space and $\ll$ the complementarily symmetric perfect L-topogenous structure on X identified with τ, by means of Proposition 7.8, and for each $\alpha \in L_0$ let $u_\alpha : X \times X \to L$ be the mappings which satisfy that $u_\alpha(x, x) = 1$ for all $x \in X$ and fulfill the following:

$$u_\alpha[f] \;=\; \begin{cases} f & \text{if } f \ll (g \wedge \overline{\alpha}) \text{ for some } g \in \tau, \\ \overline{1} & \text{otherwise.} \end{cases} \tag{7.4}$$

Lemma 7.1 *These u_α, for all $\alpha \in L_0$, satisfy the following:*

(1) $f \leq u_\alpha[f]$ *for all* $f \in L^X$,

(2) $\overline{\alpha} = u_\alpha[\overline{\alpha}]$ *for all* $\alpha \in L_0$,

(3) $u_\alpha \circ u_\alpha = u_\alpha$ *for all* $\alpha \in L_0$,

(4) $f = u_\alpha[f]$ *for all* $f \in \tau$.

Here, using Lemma 7.1, we prove this result.

Lemma 7.2 *For each* $\alpha \in L_0$ *let* $\mathcal{U}_\alpha$ *be the set of all mappings* u_α *which fulfill* (7.4) *and that* $u_\alpha(x,x) = 1$ *for all* $x \in X$. *Then the family* $(\mathcal{U}_\alpha)_{\alpha\in L_0}$ *is a family of prefilters on* $X \times X$ *and fulfills the conditions* (u1) to (u5) *of* Proposition 7.7.

Proof. For $0 < \alpha \leq \beta$ we have: (1) If $f \ll (g \wedge \overline{\alpha})$ for some $g \in \tau$, then $u_\alpha[f] = f$, and $f \ll (g \wedge \overline{\alpha}) \leq (g \wedge \overline{\beta})$ implies $f \ll (g \wedge \overline{\beta})$ which means that $u_\beta[f] = f = u_\alpha[f]$.

(2) When $f \ll (h \wedge \overline{\beta})$ for some $h \in \tau$ we get $u_\beta[f] = f \leq u_\alpha[f]$.

The other cases of f also satisfy that $u_\beta[f] \leq u_\alpha[f]$. Hence, $\mathcal{U}_\beta \subseteq \mathcal{U}_\alpha$ and (u1) is fulfilled.

From (u1) we get that $\bigcap\limits_{0<\beta<\alpha} \mathcal{U}_\beta \subseteq \mathcal{U}_\alpha$. But whenever $f \ll (g \wedge \overline{\alpha})$ for some $g \in \tau$ we have $u_\alpha[f] = f \leq u_\beta[f]$ for all $0 < \beta < \alpha$ and $\alpha = \bigvee\limits_{0<\beta<\alpha} \beta$. And also, if $f \not\ll (g \wedge \overline{\alpha})$ for all $g \in \tau$ we get that $u_\alpha[f] = \overline{1} \leq \bigcap\limits_{0<\beta<\alpha} u_\beta[f] = \overline{1}$, and then $\bigcap\limits_{0<\beta<\alpha} u_\beta[f] \geq u_\alpha[f]$ for all $f \in L^X$, which means that $\bigcap\limits_{0<\beta<\alpha} \mathcal{U}_\beta \supseteq \mathcal{U}_\alpha$. Hence $\mathcal{U}_\alpha = \bigcap\limits_{0<\beta<\alpha} \mathcal{U}_\beta$ and (u2) holds.

From that $u_\alpha[\overline{1}] = \overline{1}$ for all $\alpha \in L_0$, we get that

$$u_\alpha[\overline{1}](x) = \bigvee_{y\in X} (\overline{1}(y) \wedge u_\alpha(y,x)) = \bigvee_{y\in X} (u_\alpha(y,x)) = u_\alpha(x,x) = 1$$

for all $x \in X$, that is, $u_\alpha(x,x) \geq \alpha$ for all $\alpha \in L_0$, $x \in X$ and all $u_\alpha \in \mathcal{U}_\alpha$. Hence (u3) holds.

For $f \ll (g \wedge \overline{\alpha})$ for some $g \in \tau$ we have $f \leq \overline{\alpha}$, and then $u_\alpha[f](x) = \bigvee\limits_{y\in X} (f(y) \wedge u_\alpha(y,x)) = f(x)$ for all $x \in X$ (from that $f(x) \leq \alpha$ and $\alpha \leq u_\alpha(x,x)$). Also, when $u_\alpha[f] = \overline{1}$, we get $u_\alpha[f](x) = \bigvee\limits_{y\in X} (f(y) \wedge u_\alpha(y,x)) = u_\alpha(x,x) = 1$ for all $x \in X$.

Now, $u_\alpha^{-1}[f](x) = \bigvee_{y\in X} (f(y) \wedge u_\alpha^{-1}(y,x)) = \bigvee_{y\in X} (f(y) \wedge u_\alpha(x,y))$. If $f \ll (g \wedge \overline{\alpha})$, then $u_\alpha^{-1}[f](x) = f(x)$ for all $x \in X$, otherwise $u_\alpha^{-1}[f](x) = u_\alpha(x,x) = 1$ for all $x \in X$. That is, $u_\alpha^{-1} \in \mathcal{U}_\alpha$ whenever $u_\alpha \in \mathcal{U}_\alpha$ and therefore (u4) is fulfilled.

Since $u_\alpha \circ u_\alpha = u_\alpha$ for all $\alpha \in L_0$ and all $u_\alpha \in \mathcal{U}_\alpha$, then

$$\alpha \leq \bigvee_{u_\beta \in \mathcal{U}_\beta, u_\beta \leq u_\alpha} \beta = \bigvee_{u_\beta \in \mathcal{U}_\beta, u_\beta \circ u_\beta \leq u_\alpha} \beta,$$

and therefore (u5) is fulfilled.

Now, we prove that for all $\alpha \in L_0$, these sets $\mathcal{U}_\alpha$ are prefilters on $X \times X$.

Let $\widetilde{0} : X \times X \to L$ be the mapping defined by $\widetilde{0}(x,y) = 0$ for all $x, y \in X$. Then $\widetilde{0}[f](x) = \bigvee_{y\in X} (f(y) \wedge \widetilde{0}(y,x)) = 0$ for all $f \in L^X$ and $x \in X$, and even $\widetilde{0}(x,x) = 0 \neq 1$. Hence $\widetilde{0} \notin \mathcal{U}_\alpha$.

Let $u \in \mathcal{U}_\alpha$ and $v \geq u$. Then $v(x,x) = 1$ for all $x \in X$, and also $v[f] \geq u[f]$ for all $f \in L^X$. If $f \ll (g \wedge \overline{\alpha})$ for some $g \in \tau$, we have $f \leq \overline{\alpha}$ and $v(x,x) = 1 \geq \alpha$, and then $v[f](x) = \bigvee_{y\in X} (f(y) \wedge v(y,x)) = f(x)$ for all $x \in X$. That is, $v[f] = f$. Otherwise, if $f \ll (g \wedge \overline{\alpha})$ does not hold for all $g \in \tau$, we get that $v[f] \geq u[f] = \overline{1}$ for all $f \in L^X$. Hence $v \in \mathcal{U}_\alpha$.

Let $u, v \in \mathcal{U}_\alpha$. Then $(u \wedge v)(x,x) = u(x,x) \wedge v(x,x) = 1$ for all $x \in X$. Since

$$\begin{aligned}(u \wedge v)[f](x) &= \bigvee_{y\in X} (f(y) \wedge (u \wedge v)(y,x)) \\ &= \bigvee_{y\in X} (f(y) \wedge u(y,x)) \wedge \bigvee_{y\in X} (f(y) \wedge v(y,x)) \\ &= u[f](x) \wedge v[f](x)\end{aligned}$$

for all $f \in L^X$ and $x \in X$. Then $(u \wedge v)[f] = u[f] \wedge v[f]$ for all $f \in L^X$. If $f \ll (g \wedge \overline{\alpha})$ for some $g \in \tau$, we have $(u \wedge v)[f] = f$. Otherwise, $(u \wedge v)[f] = \overline{1}$. Hence $(u \wedge v) \in \mathcal{U}_\alpha$. Thus $(\mathcal{U}_\alpha)_{\alpha \in L_0}$ is a family of prefilters on $X \times X$ and fulfills the conditions $(u1)$ to $(u5)$. $\square$

Now, we have the following important result which shows that the L-uniform structures are compatible with the completely regular stratified L-topologies.

Proposition 7.9 *Let (X, τ) be a completely regular stratified L-topological space and Φ an L-function family of all $(\tau, \Im)$-continuous functions on X. Then the mapping $\mathcal{U} : L^{X\times X} \to L$ defined by*

$$\mathcal{U}(u) \;=\; \bigvee_{v\in\mathcal{U}_\alpha,\, v\leq u} \alpha \;\; \textit{for all } u \in L^{X\times X},$$

where $\mathcal{U}_\alpha$ is the set of all mappings u_α which fulfill (7.4) and that $u_\alpha(x, x) = 1$ for all $x \in X$, is an L-uniform structure on X compatible with τ, that is, $\tau_{\mathcal{U}} = \tau$.

Proof. From Lemma 7.2 and Proposition 7.7, we get that $\mathcal{U}$ is an L-uniform structure on X.

Now, let $g \in \tau_{\mathcal{U}}$, $g \neq \overline{1}$ and $g(x) = 1$. Then $\mathrm{int}_{\mathcal{U}}g(x) = \bigvee\limits_{u[h]\leq g} (\mathcal{U}(u) \wedge h(x)) = 1$, which means that there is some $u_\alpha \in \mathcal{U}_\alpha$ with $\mathcal{U}(u_\alpha) = 1 \geq \alpha$ such that $u_\alpha[h] = h \leq g$, $h \ll (k \wedge \overline{\alpha})$ for some $k \in \tau$, and this means that $k = (g \wedge \overline{\alpha}) \in \tau$ satisfies that $h \leq k \leq g$, $h(x) = 1$ and $k \in \tau$, that is, $k(x) = 1$, $k \leq g$ and $k \in \tau$, and then $g \in \tau$ and $\tau_{\mathcal{U}} \subseteq \tau$.

Conversely; let $g \in \tau$, $g \neq \overline{1}$ and $g \neq \mathrm{int}_{\mathcal{U}}g$, that is, there is $x \in X$ with $\mathrm{int}_{\mathcal{U}}g(x) = 0$ and $g(x) > 0$. Since $x \in s_0g \in \tau$ and (X, τ) is a completely regular space, then there exists $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in s_0g'$. Let $\mu \in L^X$ be defined by $\mu(y) = (R^1(f(y)))' = \bigvee\limits_{t\geq 1} f(y)(t)$ for all $y \in X$, then $\mu(y) \leq R_0(f(y)) \leq g(y)$ for all $y \in X$, which means that $\bigvee\limits_{t\geq 1} f(x)(t) = 1$ for all $x_1 \leq \mu$ and $\bigvee\limits_{s>0} f(y)(s) = 0$ for all $y_1 \leq g'$, that is, $f(x) = \overline{1}$ for all $x_1 \leq \mu$ and $f(y) = \overline{0}$ for all $y_1 \leq g'$, and this means that x_1, g' are Φ-separated for all $x_1 \leq \mu$, and so μ, g' are Φ-separated, and from (2.1) and Proposition 6.3 we get $\mu \ll g$, $\mu(x) = 1$.

Now, $\mathrm{int}_{\mathcal{U}}g(x) = \bigvee\limits_{u[h]\leq g} (\mathcal{U}(u) \wedge h(x)) \geq \bigvee\limits_{u_\alpha[h]\leq g} h(x)$ for some $u_\alpha \in \mathcal{U}_\alpha$ with $\mathcal{U}(u_\alpha) = 1 \geq \alpha$, which means that $\mathrm{int}_{\mathcal{U}}g(x) \geq \bigvee\limits_{h\ll g, g\in\tau} h(x)$, and is also satisfied when replacing h by μ, that is,

$$\mathrm{int}_{\mathcal{U}}g(x) \geq \bigvee_{h\ll g, g\in\tau} h(x) \geq \mu(x) = 1,$$

and then $\mathrm{int}_{\mathcal{U}} g(x) = 1 > 0$ which is a contradiction and therefore $g \in \tau_{\mathcal{U}}$. That is, $\tau = \tau_{\mathcal{U}}$, and thus $\mathcal{U}$ is compatible with τ. □

8. The relation between the $GT_{3\frac{1}{2}}$-spaces and the G-Compact spaces

Let $\mathcal{M}$ be an L-filter on a set X. The element $x \in X$ is said to be a *cluster point* of $\mathcal{M}$ if the infimum $\mathcal{M} \wedge \mathcal{N}(x)$ of $\mathcal{M}$ and the L-neighborhood filter $\mathcal{N}(x)$ at x exists. Equivalently if there exists an L-filter $\mathcal{K}$ finer than $\mathcal{M}$ which converges to x, that is, $\mathcal{K} \leq \mathcal{N}(x)$ ([13]).

G-Compact spaces. An L-topological space (X, τ) is called *G-compact* ([13]) if every L-filter on X has a cluster point in X. This notion of L-compactness fulfills the Tychonoff Theorem, that is, the product of a family of G-compact spaces is G-compact ([13]).

Proposition 8.1 [6] *Every G-compact subset of GT_2-space is closed and every G-compact GT_2-space is GT_4-space. Moreover, every closed subset of G-compact space (X, τ) is G-compact.*

Let us define the cartesian product of a number of copies of the L-unit interval I_L, equipped with the product L-topology on it, as *an L-cube.*

In the following we shall benefit from these facts.

(I) The pair (I, τ_I), of the closed unit interval I and the usual topology T_I on it, is a compact T_2-space in the classical topology.

(II) The closed unit interval $I = [0, 1]$ can be identified with the L-number $[0, 1]^{\sim}$ which has value 1 over $[0, 1]$ and 0 otherwise ([11]).

(III) The L-topology $\Im$ on I_L is, up to an identification ([11]), the usual topology on I.

Proposition 8.2 *The L-unit interval $(I_L, \Im)$ is a G-compact GT_2-space.*

Proof. Let (I, T_I) be the closed unit interval with the usual topology on it. From that in the classical topology we have (I, T_I) is a compact space, that is, every filter on I has a cluster point, then defining $\kappa : I \to I_L$ by $\kappa(r) = \tilde{r}$ for all $r \in I$, we get a homeomorphism between (I, T_I) and $(I_L, \Im)$ and hence $(I_L, \Im)$ is a G-compact space.

Also, since (I, T_I) is a T_2-space, that is, any two distinct points have disjoint T_I-neighborhoods, then using the same homeomorphism above, we have for any $f \neq g$ in I_L two disjoint $\Im$-neighborhoods, that is, $\mathcal{N}(f) \wedge \mathcal{N}(g)$ does not exist, and thus $(I_L, \Im)$ is a GT_2-space. Hence, $(I_L, \Im)$ is a G-compact GT_2-space. $\square$

Now, we prove the following result.

Proposition 8.3 *The L-unit interval $(I_L, \Im)$ is a $GT_{3\frac{1}{2}}$-space.*

Proof. Since $(I_L, \Im)$, by means of Proposition 8.2, is a G-compact GT_2-space, then from Proposition 8.1, we get that $(I_L, \Im)$ is a GT_4-space. Hence, Proposition 2.8 gives us that $(I_L, \Im)$ is a $GT_{3\frac{1}{2}}$-space. $\square$

From that G-compact spaces fulfill the Tychonoff Theorem ([13]) and from (3) in Proposition 1.3 the product L-topological space of GT_2-spaces also is a GT_2-space. Then, by means of Propositions 8.1 and 8.2, the following result goes clear.

Proposition 8.4 *The L-cube is a G-compact GT_2-space, and consequently a GT_4-space.*

Proof. Since the L-cube is the product of copies of L-unit interval $(I_L, \Im)$ and since $(I_L, \Im)$ is, by means of Proposition 8.2, G-compact GT_2-space, then from (3) in Proposition 1.3 and from that the G-compact spaces fulfill the Tychonoff Theorem it follows that the L-cube is G-compact GT_2-space. Moreover, Proposition 8.1 implies that the L-cube is GT_4-space. $\square$

We shall prove the following important result.

Proposition 8.5 *Let (X,τ) be an L-topological space and let Φ be the family of all L-continuous functions $f:(X,\tau)\to(I_L,\Im)$. For each $f\in\Phi$, let Y_f denote the space I_L. Let $Y=\prod\limits_{f\in\Phi}Y_f$ with the product L-topology $\Im_Y$ on it. If (X,τ) is a $GT_{3\frac{1}{2}}$-space, then X is homeomorphic to a subspace of Y. More Precisely, the mapping $e:X\to Y$, $e(x)=\hat{x}=\prod\limits_{f\in\Phi}x_f$, $x_f=f(x)$, is a homeomorphism from X onto $e(X)$ when (X,τ) is a $GT_{3\frac{1}{2}}$-space.*

Proof. Suppose that (X,τ) is a $GT_{3\frac{1}{2}}$-space and consider the evaluation map $e:X\to Y$, $x\mapsto(f(x))_{f\in\Phi}=\hat{x}$. In view of Corollary 6.1, e is one - one. Also e is L-continuous from that every $f\in\Phi$ is continuous (each composition $p_f\circ e:X\to Y_f$, $x\mapsto f(x)$ is continuous, where $p_f:Y\to Y_f$ denotes the projection map). If $Z=e(X)$, then $e:(X,\tau)\to(Z,\Im_Z)$ is a bijection L-continuous mapping.

Now we show that e is L-open. As in the proof of Proposition 6.2, the family

$$B=\{f^{-1}(\mu)\mid f\in\Phi,\mu\in\Im\}.$$

is a base for the L-topology τ on X. Since for a family $(g_j)_{j\in J}$ of L-sets in X, we have $e(\bigvee\limits_{j\in J}g_j)=\bigvee\limits_{j\in J}e(g_j)$ and $e(g_1\wedge\cdots\wedge g_n)=e(g_1)\wedge\cdots\wedge e(g_n)$, it follows that to show that e is L-open, it is sufficient to show that $e(\rho)\in\Im_Z$ for each $\rho\in B$.

Let $f\in\Phi$, $\mu\in\Im$ in Y_f and $\rho=f^{-1}(\mu)=\mu\circ f$ with $\rho\in B$. Then

$$e(\rho)(\hat{x})=\bigvee_{x\in e^{-1}(\hat{x})}\rho(x)=\rho(x)=\mu(f(x))=\bigvee_{\hat{x}\in p_f^{-1}(f(x))}\mu(f(x))=p_f^{-1}(\mu)(\hat{x})$$

for all $\hat{x}=e(x)\in Z$. Since $p_f^{-1}(\mu)\mid_Z=e(\rho)$ and p_f is continuous, then $p_f^{-1}(\mu)\mid_Z$ is open in τ_Z and thus $e(\rho)\in\tau_Z$. Hence e is L-open and therefore $e:X\to Z$ is a homeomorphism and (X,τ) is homeomorphic to a subspace of $Y=\prod\limits_{f\in\Phi}Y_f$. $\square$

The following result now is obtained.

Proposition 8.6 *Let (X,τ) be an L-topological space. Then (X,τ) is a $GT_{3\frac{1}{2}}$-space* if and only if X *is homeomorphic to a subspace of an L-cube.*

Proof. From Proposition 8.5, the necessity of the condition follows.

Since the L-unit interval $(I_L, \Im)$ is a $GT_{3\frac{1}{2}}$-space from Proposition 8.3 and that the product L-topological space of $GT_{3\frac{1}{2}}$-spaces is also a $GT_{3\frac{1}{2}}$-space from Corollary 3.1, then (X, τ) itself is a $GT_{3\frac{1}{2}}$-space. □

The following results come easily.

Proposition 8.7 *A G-compact space (X, τ) is a GT_2-space* if and only if *it is a $GT_{3\frac{1}{2}}$-space.*

Proof. Since any GT_2-space (X, τ) which is G-compact is a GT_4-space, then it is, by means of Proposition 2.8, a $GT_{3\frac{1}{2}}$-space.

The other direction follows from Proposition 2.1 and from (1) in Proposition 1.3. □

Lemma 8.1 [6] *If τ_1 and τ_2 are L-topologies on a set X, τ_1 is finer than τ_2 and (X, τ_1) is G-compact, then (X, τ_2) also is G-compact.*

Now we prove this result.

Proposition 8.8 *Let τ_1 and τ_2 be L-topologies on a set X with τ_1 be finer than τ_2, and let (X, τ_1) be a G-compact space and (X, τ_2) be a $GT_{3\frac{1}{2}}$-space. Then τ_1 is equivalent to τ_2.*

Proof. From Proposition 2.13, we get (X, τ_1) is also a $GT_{3\frac{1}{2}}$-space, and from Lemma 8.1 we have (X, τ_2) is also a G-compact space. Then we can find the identity mapping $\mathrm{id}_X : (X, \tau_1) \to (X, \tau_2)$ which is a bijective L-continuous mapping and L-open, that is, a homeomorphism. Hence, (X, τ_1) is equivalent to (X, τ_2). □

Now, we show this essential proposition.

Proposition 8.9 *For every L-topological space (X, τ) the following are equivalent:*

(1) *(X, τ) is a $GT_{3\frac{1}{2}}$-space,*

(2) *X is homeomorphic to a subspace of an L-cube,*

(3) *X is homeomorphic to a subspace of a G-compact GT_2-space,*

(4) *X is homeomorphic to a subspace of a GT_4-space.*

Proof.

(1) $\Rightarrow$ (2): Follows from Proposition 8.6.

(2) $\Rightarrow$ (3): Since every L-cube is a G-compact GT_2-space, then this implication is true.

(3) $\Rightarrow$ (4): Follows from that every G-compact GT_2-space is a GT_4-space.

(4) $\Rightarrow$ (1): From (3) in Proposition 1.3 we have that every subspace of a GT_4-space is a GT_4-space, and therefore is a $GT_{3\frac{1}{2}}$-space. $\Box$

References

[1] J. Adámek, H. Herrlich and G. Strecker; *Abstract and Concrete Categories*, John Wiley & Sons, Inc. New York et al., 1990.

[2] F. Bayoumi and I. Ibedou; *T_i-spaces*, I, J. of the Egyptian Mathematical Society, Vol. **10** (**2**) (2002) 179 - 199.

[3] F. Bayoumi and I. Ibedou; *T_i-spaces*, II, J. of the Egyptian Mathematical Society, Vol. **10** (**2**) (2002) 201 - 215.

[4] F. Bayoumi and I. Ibedou; *On GT_i-spaces*, J. of Inst. of Math. and Comp. Sci., Vol. **14**, No. **3**, (2001) 187 - 199.

[5] F. Bayoumi; *On initial and final fuzzy uniform structures*, Fuzzy Sets and Systems, Vol. **133**, Issue **3** (2003) 99 - 319.

[6] F. Bayoumi and I. Ibedou; *The relation between the GT_i-spaces and fuzzy proximity spaces, G-compact spaces, fuzzy uniform spaces*, The Journal of Chaos, Solitons and Fractals, **20** (2004) 955 - 966.

[7] F. Bayoumi and I. Ibedou; *$GT_{3\frac{1}{2}}$-spaces*, I, J. of the Egyptian Mathematical Society, Submitted.

[8] C. L. Chang; *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182 - 190.

[9] Á. Császár; *General Topology*, Akadémiai Kiadó, Budapest 1978.

[10] P. Eklund and W. Gähler; *Fuzzy filter functors and convergence*, in: Applications of Category Theory To Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht et al., (1992), 109 - 136.

[11] S. Gähler and W. Gähler; *Fuzzy real numbers*, Fuzzy Sets and Systems, **66** (1994) 137 - 158.

[12] W. Gähler; *The general fuzzy filter approach to fuzzy topology*, I, Fuzzy Sets and Systems, **76** (1995) 205 - 224.

[13] W. Gähler; *The general fuzzy filter approach to fuzzy topology*, II, Fuzzy Sets and Systems, **76** (1995) 225 - 246.

[14] W. Gähler, F. Bayoumi, A. Kandil and A. Nouh; *The theory of global fuzzy neighborhood structures.* II, *Fuzzy topogenous orders*, Fuzzy Sets and Systems, **98** (1998) 153 - 173.

[15] W. Gähler, F. Bayoumi, A. Kandil and A. Nouh; *The theory of global fuzzy neighborhood structures.* III, *Fuzzy uniform structures*, Fuzzy Sets and Systems, **98** (1998) 175 - 199.

[16] J. A. Goguen; *L-fuzzy sets*, J. Math. Anal. Appl., **18** (1967) 145 - 174.

[17] B. Hutton; *Uniformities on fuzzy topological spaces*, J. Math. Anal. Appl., **58** (1977) 559 - 571.

[18] A. Kandil and M. E. EL-Shafee; *Regularity axioms in fuzzy topological spaces and FR_i-Proximities*, Fuzzy Sets and Systems, **27** (1988) 217 - 231.

[19] A. Kandil, F. Bayoumi and A. Nouh; *On strong notions in fuzzy topological spaces*, The Journal of Fuzzy Mathematics, Vol. **6**, No. **3**, (1998) 751 - 773.

[20] A. K. Katsaras; *Fuzzy proximity spaces*, J. Math. Anal. Appl., **68** (1979) 100 - 110.

[21] A. K. Katsaras; *Fuzzy proximities and fuzzy completely regular spaces*, I, An. Ştiinto Univ. Iasşi Sect. Ia Math. (N.S.) **26** (1980) 31 - 41.

[22] A. K. Katsaras and C. G. Petalas; *On fuzzy syntopogenous structures*, J. Math. Anal. Appl., **99** (1984) 219 - 236.

[23] R. Lowen; *Initial and final fuzzy topologies and fuzzy Tychonoff theorem*, J. Math. Anal. Appl., **58** (1977) 11 - 21.

[24] R. Lowen; *A comparison of different compactness notions in fuzzy topological spaces*, J. Math. Anal. Appl., **64** (1978) 446 - 454.